

## **THE DISCRETE ORDINATES METHOD IN CYLINDRICAL COORDINATES: A NEW SOLUTION PROCEDURE BASED ON THE EXISTENCE OF SETS OF INDEPENDENT DIRECTIONS**

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### **RESUMEN**

El fenómeno de Intercambio de Energía Radiante se encuentra presente en una gran variedad de problemas de ingeniería que van desde el diseño de cámaras de combustión hasta aplicaciones de percepción remota desde satélites, pasando por estudios de confort arquitectónico. Debido a sus muchas ventajas, el Método de las Ordenadas Discretas (MOD) se ha convertido en el procedimiento más utilizado para resolver la Ecuación de Intercambio Radiante (EIR).

La necesidad de resolver problemas cada vez mayores y de requisitos de cálculo más rápidos ha resultado en la necesidad de desarrollar algoritmos más rápidos y de menor demanda de memoria. Como consecuencia de esto, algoritmos paralelos ejecutándose en máquinas acordes, se han convertido en herramientas comunes para la comunidad de ingeniería. El MOD no ha escapado este reto y códigos paralelos de este método son cada vez más frecuentes. Sin embargo, cuando es aplicado a sistemas en coordenadas cilíndricas, el MOD resulta altamente interdependiente y recursivo por lo que no se presta a ningún esquema de paralelización práctico. Esta limitación es de particular importancia en Esquemas de Paralelización basados en el Pase de Mensajes (EPPM) ejecutados sobre Máquinas de Memoria Distribuida (MMD).

En este trabajo se presenta un Nuevo Procedimiento de Solución (NPS) basado en Grupos de Direcciones Independientes (GDI) encontrados en la formulación del MOD en coordenadas cilíndricas. Estos GDI pueden ser resueltos, prácticamente, con total independencia entre sí, por lo que tanto la paralelización, como el balance de cargas entre procesadores son tareas posibles. Además, los GDI siguen un patrón claro que permite su especificación inmediata para cualquier orden de cuadratura (S-N).

### **ABSTRACT**

Radiative Transfer is present in a wide variety of engineering applications ranging from furnace design and combustion analysis to architectural comfort studies and satellite remote sensing. Because of its many advantages, the Discrete Ordinates Method (DOM) has become the most widely used procedure for the solution of the Radiative Transfer Equation (RTE).

The need to solve bigger and more calculation-intensive problems has resulted in the need for faster and less memory demanding procedures. As a result of this, parallel algorithms running on parallel machines have become a common tool for the engineering community. The DOM has not escaped this challenge and parallelized version of the DOM are increasingly

common. However, when applied to cylindrical coordinate systems, the DOM becomes a very interdependent procedure that does not allow for any practical parallelization scheme. This shortcoming is particularly noticeable in Message Passing Parallelizing Schemes (MPPS) over Distributed Memory Machines (DMM).

In this paper, a new solution procedure based on the existence of Sets of Independent Directions (SID) in the formulation of the DOM in cylindrical coordinates is presented. These SID can be, basically, solved independently from each other, allowing therefore, for the parallelization of the algorithm and for a balanced distribution of work among different processors. Furthermore, the SID follows a clear pattern and its specification for any order of quadrature (S-N) is clear and straight forward.

## INTRODUCTION

Radiative Transfer is present in a wide variety of engineering applications ranging from furnace design and combustion analysis to architectural comfort studies and satellite remote sensing [1]. In these applications, the Radiative Transfer Equation (RTE), has to be solved. Because of its many advantages, the Discrete Ordinates Method (DOM) [2] has become the most widely used procedure –almost standard– for the solution of the RTE.

The need to solve bigger and more calculation-intensive problems has resulted in the recent development of parallel algorithms of the DOM [3,4,5,6]. These parallel algorithms are, normally, modifications of previously existing serial implementations. However, the Standard Solution Procedure (SSP) normally suggested in the literature for serial applications of the DOM to cylindrical coordinate systems, is based on quadrants [7,8]. This approach becomes a very interdependent procedure that does not allow for any practical parallelization scheme when working with Message Passing Parallelizing Schemes (MPPS) over Distributed Memory Machines (DMM).

In this paper, a New Solution Procedure (NSP) based on the existence of Sets of Independent Directions (SID) in the formulation of the DOM in cylindrical coordinates is presented. These SID can be, basically, solved independently from each other allowing, therefore, for the parallelization of the algorithm and for a balanced distribution of work among different processors.

The general mathematical model for radiative exchange in cylindrical coordinates is presented in the next section. Then, the Discrete Ordinates Method, as it applies to the problem under consideration, is briefly reviewed and is followed by the description of the NSP and the SID. Finally, some conclusions are drawn about the utility of the NSP and the SID in the development of parallel algorithms.

## MATHEMATICAL MODEL

The radiative transfer equation (RTE), describing the propagation of monochromatic radiation intensity ( $I$ ) along a line of sight ( $\zeta$ ), can be written as:

$$\frac{dI(\zeta, \omega)}{d\zeta} = -(a+s)I(\zeta) + aI_b(\zeta, T) + \frac{s}{4\pi} \int_0^{4\pi} I(\zeta, \omega_i) \Phi(\omega, \omega_i) d\omega_i \quad (1)$$

where  $a$  and  $s$  are the monochromatic absorption coefficient and the scattering coefficient, respectively.  $I_b$  is the black body emitted intensity;  $\Phi$  is the phase function;  $T$  represents temperature; and  $\omega$  is the solid angle.

Equation 1 is subjected (for diffuse surfaces) to the following boundary conditions:

$$I^\dagger(\zeta, \omega) = \varepsilon I_b^\dagger + \frac{\psi}{\pi} \int_0^{2\pi} \Gamma(\zeta, \omega_j) \alpha \, d\omega_j \quad (2)$$

In Equation 2, superscripts +, and - indicate radiation going from the boundary toward the medium inside the domain, and from the medium toward the boundary respectively.  $\alpha$  represents the cosine of the angle between the direction of propagation of  $I_\lambda$  and the normal to the given boundary. The reflectance is denoted by  $\psi$ , and is understood to be monochromatic.

The problem of interest in this study is that of two-dimensional, cylindrically axisymmetric geometries as depicted in Figure 1.

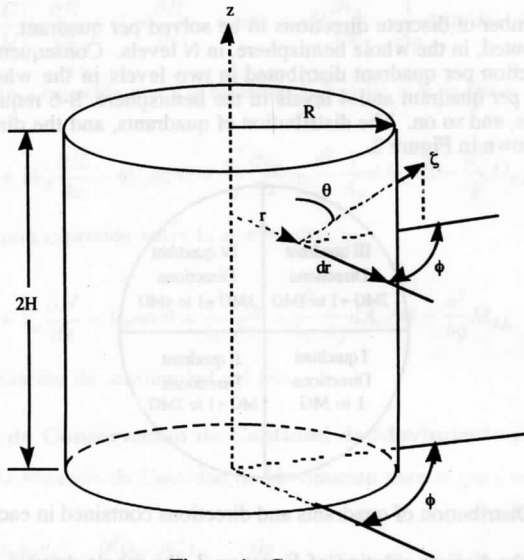


Figure 1. Geometry

When applied to a discrete set of directions in this geometry, Equations 1 and 2 become [7, 9]:

$$\frac{\mu_m}{r} \frac{\partial [r I_m]}{\partial r} - \frac{1}{r} \frac{\partial [\eta_m I_m]}{\partial \phi} + \xi_m \frac{\partial I_m}{\partial z} = - (a+s) I_m + a I_b(T) + \frac{s}{4\pi} \sum_{m^i} W_{m^i} I_{m^i} \Phi(m, m^i) \quad (3)$$

$$I_m^\dagger = \varepsilon I_b^\dagger + \frac{\psi}{\pi} \sum_{m^i} W_{m^i} \mu_{m^i} I_{m^i} \quad \text{at } r = R \quad (4)$$

where  $0 \leq r \leq R$ ,  $R$  being the cylinder radius;  $I_m = I(r, \omega)$ ;  $I_{m^i} = I(r, \omega_j)$ ;  $m$  and  $m^i$  represent discrete directions;  $\mu_m$ ,  $\eta_m$ , and  $\xi_m$  are the direction cosines for direction  $m$ ;  $W_{m^i}$  and  $\mu_{m^i}$  are the weight and direction cosine for direction  $m^i$ , respectively; and  $\Phi(m, m^i)$  is the discretized

scattering phase function between directions  $m$  and  $m^i$ . The boundary at  $r = 0$  is treated as a specular boundary.

**THE DISCRETE ORDINATES METHOD (S-N)**

The Discrete Ordinates Method (DOM or S-N) consists in the selection of an appropriate set of directions on which to solve Equations 3 and 4. The procedures for selecting the quadrature of integration (directions), obtaining the direction cosines and associated weights, or for evaluating the discretized phase function are well described elsewhere [1,2,7,9,10]. For the symmetric problem being treated here, the order of the quadrature is defined by the value of  $N$  according to the following expression:

$$MG = \frac{N(N+2)}{8} \tag{5}$$

where  $MG$  is the number of discrete directions to be solved per quadrant. Furthermore, all the directions are distributed, in the whole hemisphere, in  $N$  levels. Consequently, S-2 requires the solution of one direction per quadrant distributed in two levels in the whole hemisphere, S-4 requires 3 directions per quadrant and 4 levels in the hemisphere, S-6 requires 6 directions per quadrant in six levels, and so on. The distribution of quadrants, and the directions contained in each quadrant are shown in Figure 2.

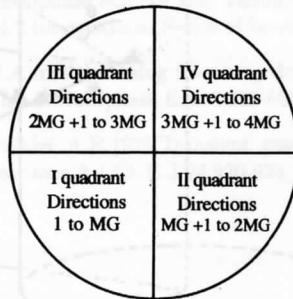


Figure 2. Distribution of quadrants and directions contained in each quadrant

In order to find the discrete solution of Equation 3, the whole domain is subdivided into a series of  $MR \times MZ$  control volumes –not necessarily equal– as depicted in Figure 3a.

After multiplying Equation 3 by  $dv = 2\pi r dr dz$ , and integrating over a given control volume the following expression is obtained:

$$I_m^p = \frac{|\mu_m| A_1 I_m^\pi + |\xi_m| B_1 I_m^z + C_1 I_{m-1/2}^p + \beta S_m^p}{(|\mu_m| A_2 + |\xi_m| B_2 + C_2) + \beta} \tag{6}$$

in this expression, which is solved for each direction ( $m$ ) and for each control volume, superscript  $p$  indicates average over the control volume;  $I_m^\pi$  and  $I_m^z$  are the reference (known) intensities in the radial and axial directions respectively (see Figure 3b for an example);  $S_m^p$ , the source, is equal to the last two terms on the right-hand side of Equation 3; and  $\beta$  is the extinction coefficient ( $\beta = a + s$ ). Also in Equation 6 :  $A_1, A_2, B_1, B_2, C_1,$  and  $C_2$  are, for each control volume, constants that depend on the geometry, the direction being considered, and the local

extinction coefficient. These constants can be evaluated in a deterministic manner for each particular problem [7,11] and are not relevant to this study. Of particular importance to this work is the term  $I_{m-1/2}^P$  to be explained later.

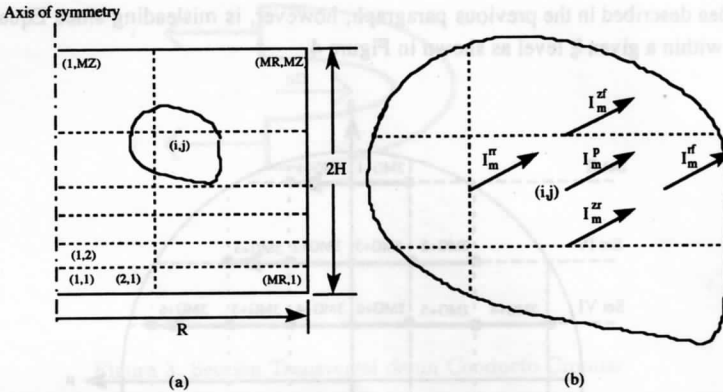


Figure 3. (a) Discretized domain. (b) Example solution for a direction in the fourth quadrant

After Equation 6 has been solved for the average intensity in the control volume ( $I_m^P$ ) for the given direction (m), the final (unknown) intensities ( $I_m^{rf}$ ,  $I_m^{zf}$ , and  $I_{m+1/2}^P$ ) can be found from the expressions:

$$I_m^{rf} = \frac{I_m^P}{aa} - [(1 - aa)] I_m^{pi} \tag{7a}$$

$$I_m^{zf} = \frac{I_m^P}{bb} - [(1 - bb)] I_m^{zr} \tag{7b}$$

$$I_{m+1/2}^P = \frac{I_m^P}{cc} - [(1 - cc)] I_{m-1/2}^P \tag{7c}$$

where aa, bb, and cc are constants that can be evaluated in a deterministic manner based on the geometry of the control volume, the direction under consideration and the local extinction coefficient [7,11].

Equations 6, 7a, and 7b clearly indicate a marching or sequential solution along the discretized domain. Equations 6 and 7c, on the other hand, seem to indicate –misleadingly– a marching solution along the directions of quadrature.

### THE SETS OF INDEPENDENT DIRECTIONS

Equation 7c suggests a marching solution for the  $I_{m+1/2}^P$  terms. With  $I_{1-1/2}^P$  known  $I_{1+1/2}^P$  could be evaluated, this in turn would lead to  $I_{2+1/2}^P$  and so on until  $I_{4M+1/2}^P$  is finally calculated. Because of this apparent marching dependency, the standard solution procedure (SSP) normally suggested in the literature for Equation 6 [7, 8] is based on quadrants as follows: starting from the upper right corner the whole domain is covered (solving Equations 6

and 7) for directions 1 to MG (first quadrant) until the lower left corner is reached, next the domain is solved from the lower right corner to the upper left for directions 2MG+1 to 3MG (third quadrant), then the procedure continues from the upper left corner to the lower right (second quadrant –directions MG+1 to 2MG–), and finally the domain is solved from lower left to upper right corners (fourth quadrant –directions 3MG+1 to 4MG–).

The idea described in the previous paragraph, however, is misleading since Equation 7c is only valid within a given  $\xi$  level as shown in Figure 4.

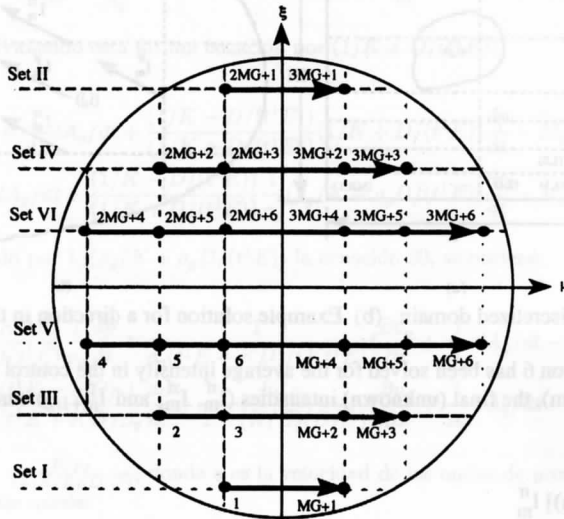


Figure 4. Sets of independent directions for an S-6 implementation of the DOM

In Figure 4, each arrow indicates a level and the direction for the marching procedure of Equation 7c. In general, a given S-N implementation of the DOM has N  $\xi$  levels or SIDs. The smallest set includes two directions (sets I and II) and the largest set includes N directions as shown in Table I.

It should be noticed that all sets contain mirroring directions in two different quadrants. For example, Set III contains mirroring directions in the first and second quadrant while Set IV includes mirroring directions in the third and fourth quadrant. In general, odd numbered sets relate first and second quadrants, while even numbered sets relate the third and the fourth quadrant.

Based on the previous description of SIDs, a new solution procedure (NSP) for Equations 6 and 7 can be proposed as follows:

- 1 The domain is divided into MRxMZ control volumes and the order of quadrature (N) is selected.
- 2 The solution (Equations 6 and 7) proceeds by sets, starting with Set I and ending with Set N.
- 3 For odd numbered sets, the domain is swept first from the upper right corner to the lower left corner (first quadrant) and then from the upper left corner to the lower right corner (second quadrant).

- 4 For even numbered sets, the domain is swept first from the lower right corner to the upper left corner (third quadrant) and then from the lower left corner to the upper right corner (fourth quadrant).
- 5 In all cases, and for the first direction (m) of each set,  $I_{m-1/2}^P = I_m^P$

Table 1. Sets of Independent Directions

N	Sets	Directions
≥ 2	I	1 ⇒ MG+1
	II	2MG+1 ⇒ 3MG+1
≥ 4	III	2,3 ⇒ MG+2, MG+3
	IV	2MG+2, 2MG+3 ⇒ 3MG+2, 3MG+3
≥ 6	V	4,5,6 ⇒ MG+4, MG+5, MG+6
	VI	2MG+4, 2MG+5, 2MG+6 ⇒ 3MG+4, 3MG+5, 3MG+6
⋮	⋮	⋮
= K	K-1	MG-K/2+1, MG-K/2+2, ..., MG ⇒ 2MG-K/2+1, 2MG-K/2+2, ..., 2MG
	K	3MG-K/2+1, 3MG-K/2+2, ..., 3MG ⇒ 4MG-K/2+1, 4MG-K/2+2, ..., 4MG

This NSP allows the application of Equation 7c in a straight forward manner within each level and avoids the need for storing intermediate values of  $I_{m-1/2}^P$ . Further, since each set contains mirroring directions between two adjacent quadrants, the symmetric left boundary of the domain can be updated instantaneously while solving for each individual set.

Finally, it should be pointed out that if the resulting algorithm is to be parallelized based on Message Passing Parallelizing Schemes (MPPS) over Distributed Memory Machines (DMM) then, because of its minimum requirement for message passing among sets, clearly, the NSP lends itself to a parallelization scheme based on sets. Moreover, knowing the number of directions to be solved in each set, the load distribution among processors can be easily optimized.

**CONCLUSIONS**

In this paper, a new solution procedure for the Discrete Ordinates Method in cylindrical coordinates has been presented. The procedure is based on sets of independent directions. For a given S-N implementation of the DOM, N sets of independent directions, distributed in N ξ levels are possible. The smaller sets include two directions and the larger sets include N directions.

The proposed solution procedure as well as the concept of sets of independent directions are particularly important for parallelization strategies based on message passing libraries. In particular, minimizing message passing between processors and optimizing work-load-distribution among the same processors are, now, possible tasks.

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